Functor-of-points formalism

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Recollections on functor of points formalism:

Yoneda $\Rightarrow$ $\quad \mathrm{Scheme} \hookrightarrow \mathrm{Fun}(\mathrm{Scheme}^{op}, \mathrm{Sets})$

$$X \longmapsto \mathrm{Map}(-, X)$$

<u>Site:</u> Category (assume w/ fiber products) and covering families satisying locality, stability under base change, identity

<u>Ex:</u> 1) open sets in a topological space — small

2) local homeomorphisms and top spaces, manifolds smooth or cplx

3) Sch /k w/ Zarski

$\cap$

étale

$\cap$

fppf topology

big

fppf topology

4) Ring^{op}/k w/ same topologies

5) same as 3 or 4 but with finite type requirement

Given a covering $\{U_\alpha \to X\}$ a functor is a sheaf if (satisfies descent) if

$$F(X) \longrightarrow \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$$

is an equalizer.

Lem: The restriction functor τ = etale, fppf, Zariski

$$\mathrm{Sh}(\mathrm{Sch}/k)_\tau \longrightarrow \mathrm{Sh}(\mathrm{Ring}^{op}/k)_\tau$$

Is an equivalence

"different sites, equivalent topos"

Notation: X is a scheme, $X(R) = \mathrm{Map}(\mathrm{Spec}\,R, X)$ is functor (covariant)

$$\mathrm{Ring}/k \longrightarrow \mathrm{Set}$$

Ex: Group scheme $\mathbb{G}_m$

$$\mathbb{G}_m(R) = \mathrm{Map}(k[t^{\pm}], R)$$
$$= R^{\times} \text{ as a set} \rightsquigarrow \text{clearly a gp.}$$

$$GL_n(R) = GL_n(R)$$
$$= \left(\mathrm{End}_R(R^{\oplus n})\right)^{\times}$$

Useful fact: A functor s.t. filtered (e.g. directed) systems of rings R_i

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$$\operatorname*{colim}_{i\in I} F(R_i) \xrightarrow{\cong} F(\operatorname*{colim}_{i\in I} R_i)$$

is called "locally finitely presented" (LFP)

- Restriction $Sh(Ring^{op}_{/k}) \leftrightarrows Sh(Ring^{op}_{ft/k})$ has a fully-faithful left adjoint whose essential image consists of LFP sheaves
  - ↳ LFP sheaf is functorially determ. by its restriction to f.t! rings/k

- A scheme is LFP $\Leftrightarrow$ its functor of points is LFP